

Controller Design for Unstable Distributed Plants

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Abstract. In this paper we solve the H_∞ mixed sensitivity minimization problem for a class of unstable distributed systems. The solution is based on an extension of the skew Toeplitz methodology developed in [1], [3], [4], [6], [10]. The key mathematical fact used is that the skew Toeplitz operators arising in the unstable case are finite rank perturbations of the classical skew Toeplitz operators obtained from compressions of rational functions.

1 Introduction

The purpose of this paper is to solve the mixed sensitivity H^∞ -optimization problem for distributed plants with a finite number of unstable poles. In the previous theory developed in [1], [2], [3], [4], [6], [10], [12] for stable distributed (or arbitrary lumped) plants, the computational procedure involves computing the singular values and vectors of a certain class of skew Toeplitz operators.

The methods given in the above papers require that the corresponding skew Toeplitz operators take a special form which is not satisfied in the unstable distributed parameter system case. In this paper, we develop a technique which is valid in the more general unstable case. This development will be carried out in the mixed sensitivity (two block) framework. As in the previous work, the computation of the optimal performance and corresponding optimal controller will be reduced to a finite dimensional matrix problem. In the stable case the size of the matrix only depends on the McMillan degree of the weighting filters. In the case of unstable plants, the size of the corresponding matrix will be seen to depend on the number of right half plane poles of the plant as well. The dimension of this matrix can be computed *a priori*. The key mathematical fact that we use is that the skew Toeplitz operators obtained in the unstable case are finite rank perturbations of the classical skew Toeplitz operators obtained from compressions of rational functions.

The full version of the present paper appears in [9]. We would like to point out that the skew Toeplitz techniques

employed here have been used to synthesize controllers for several types of flexible structures and delay systems in [7].

In section 2 we will show how two types of 2-block H^∞ -minimization problems for unstable plants reduce to a standard form involving a skew Toeplitz operator. In section 3 we will obtain a linear system of equations (the *singular system*) from which one can compute the singular values and vectors of this class of operators. In section 4 we work out an example.

2 Mixed sensitivity problems with unstable plant

In this section we will show that several 2-block H^∞ -minimization problems reduce to the computation of the norm of a certain skew Toeplitz operator. We begin with some notation. The Hardy spaces H^2 and H^∞ are defined on the unit disc in the standard way. We denote

$$\begin{aligned}\tilde{H}^\infty &:= \{f \in H^\infty : \overline{f(\bar{z})} = f(z)\} \\ R\tilde{H}^\infty &:= \{\text{rational functions in } \tilde{H}^\infty\}\end{aligned}$$

We consider the feedback configuration of Fig. 1 with

$$P = \frac{G_n}{G_d}$$

and $G_n \in \tilde{H}^\infty$, $G_d \in R\tilde{H}^\infty$. We assume that (i) $G_n = m_n G_{no}$, where $m_n \in \tilde{H}^\infty$ is inner (arbitrary) and $G_{no} \in R\tilde{H}^\infty$ is outer, and (ii) G_n , G_d have no common zeros in the closed unit disc. We also write $G_d = m_d G_{do}$ where $m_d \in R\tilde{H}^\infty$ is inner and $G_{do} \in \tilde{H}^\infty$ is outer. Under these assumptions there exist $X \in R\tilde{H}^\infty$ and $Y \in \tilde{H}^\infty$ such that

$$XG_n + YG_d = 1. \quad (1)$$

(To construct solutions of (1), X must be chosen to satisfy a set of interpolation constraints at the zeros of G_d in the closed unit disc so that $Y = (1 - XG_n)/G_d$ belongs to \tilde{H}^∞ . Since the constraints are finite in number, X can always

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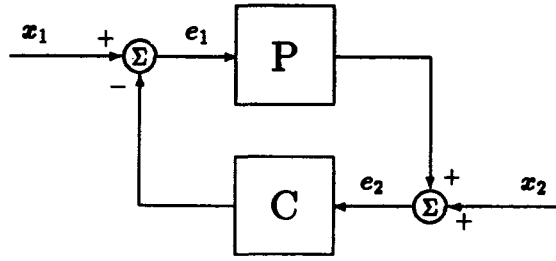


Figure 1: Standard feedback configuration

be chosen to be rational.) The set of all controllers which stabilize the plant can now be written in the form

$$C = \frac{X + QG_d}{Y - QG_n}$$

for some $Q \in \tilde{H}^\infty$. Now let $S := (1 + PC)^{-1}$ and note that

$$S = 1 - XG_n - QG_nG_d. \quad (2)$$

We first consider the following problem. Find

$$\mu = \inf_{\text{stabilizing } C} \left\| \begin{bmatrix} W_1 S \\ W_2 (S - 1) \end{bmatrix} \right\|$$

where $W_1, W_2 \in R\tilde{H}^\infty$ are given weighting functions with $W_1^{-1}, W_2^{-1} \in R\tilde{H}^\infty$. From (2) we can write

$$\mu = \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_1 - W_1 X G_n \\ -W_2 X G_n \end{bmatrix} - \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} Q G_n G_d \right\|_\infty.$$

Let $W_1^* W_1 + W_2^* W_2 = G^* G$ where $G, G^{-1} \in R\tilde{H}^\infty$. Then

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} W_1 G^{-1} \\ W_2 G^{-1} \end{bmatrix} G$$

is an inner-outer factorisation. Moreover

$$L := \begin{bmatrix} W_1^* G^{-*} & W_2^* G^{-*} \\ -W_2^* G^{-1} & W_1^* G^{-1} \end{bmatrix}$$

is square and unitary. Hence

$$\begin{aligned} \mu &= \inf_{Q \in \tilde{H}^\infty} \left\| L \left(\begin{bmatrix} W_1 \\ 0 \end{bmatrix} - \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} X G_n - \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} Q G_n G_d \right) \right\|_\infty \\ &= \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} G^{-*} W_1^* W_1 - X G G_n - Q G G_n G_d \\ -W_1 W_2 G^{-1} \end{bmatrix} \right\|_\infty. \end{aligned}$$

Since $G^{-*} W_1^* W_1 \in RL^\infty$, there exists a finite Blaschke product $b_1 \in R\tilde{H}^\infty$ such that $W_0 := b_1 G^{-*} W_1^* W_1$ and belongs to $R\tilde{H}^\infty$. Thus

$$\mu = \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_0 - G X b_1 G_n - Q b_1 G G_n G_d \\ -W_1 W_2 G^{-1} \end{bmatrix} \right\|_\infty. \quad (3)$$

We now write

$$\begin{aligned} \tilde{W}_0 &:= G X G_n \\ m &:= b_1 m_n \\ m_n &:= b_1 m_n m_d \\ G_0 &:= -W_1 W_2 G^{-1}. \end{aligned}$$

Then (3) reduces to

$$\mu = \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_0 - \tilde{W}_0 m - Q m_n G G_n G_d \\ G_0 \end{bmatrix} \right\|_\infty.$$

Now let

$$Q_1 := Q G G_n G_d.$$

Then, under mild conditions on the plant and the weighting functions, it can be shown that

$$\mu = \inf_{Q_1 \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_0 - \tilde{W}_0 m - Q_1 m_n \\ G_0 \end{bmatrix} \right\|_\infty$$

We can do a similar type of reduction for the following 2-block minimization problem. Find

$$\mu = \inf_{\text{stabilizing } C} \left\| \begin{bmatrix} W_1 S \\ W_2 C S \end{bmatrix} \right\| \quad (4)$$

where $W_1, W_2 \in R\tilde{H}^\infty$ are given weighting functions with $W_1^{-1}, W_2^{-1} \in R\tilde{H}^\infty$. Since $CS = XG_d + QG_d^2$ we can write

$$\mu = \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_1 \\ 0 \end{bmatrix} - \begin{bmatrix} W_1 G_n \\ -W_2 G_d \end{bmatrix} X - \begin{bmatrix} W_1 G_n \\ -W_2 G_d \end{bmatrix} Q G_d \right\|_\infty.$$

Since $G_n^* G_n$ is rational, $W_1^* G_n^* G_n W_1 + W_2^* G_d^* G_d W_2 = G^* G$ for some $G, G^{-1} \in R\tilde{H}^\infty$. Then

$$\begin{bmatrix} W_1 G_n \\ -W_2 G_d \end{bmatrix} = \begin{bmatrix} W_1 G_n G^{-1} \\ -W_2 G_d G^{-1} \end{bmatrix} G$$

is an inner-outer factorization. Next note that the matrix

$$L := \begin{bmatrix} W_1^* G_n^* G^{-*} & -W_2^* G_d^* G^{-*} \\ W_2^* G_d G^{-1} & W_1^* G_n G^{-1} \end{bmatrix}$$

is square and unitary. Hence

$$\begin{aligned} \mu &= \inf_{Q \in \tilde{H}^\infty} \left\| L \left(\begin{bmatrix} W_1 \\ 0 \end{bmatrix} - \begin{bmatrix} W_1 G_n \\ -W_2 G_d \end{bmatrix} X - \begin{bmatrix} W_1 G_n \\ -W_2 G_d \end{bmatrix} Q G_d \right) \right\|_\infty \\ &= \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_1^* W_1 G_n^* G^{-*} - X G - Q G G_d \\ W_1 W_2 G_d G^{-1} \end{bmatrix} \right\|_\infty \end{aligned} \quad (5)$$

Again, there exists a finite Blaschke product $b_1 \in R\tilde{H}^\infty$ such that $W_0 := b_1 m_n W_1^* W_1 G_n^* G^{-*}$ belong to $R\tilde{H}^\infty$. Define

$$\begin{aligned} \tilde{W}_0 &:= G X \\ m &:= b_1 m_n \\ m_n &:= b_1 m_n m_d \\ G_0 &:= W_1 W_2 G_d G^{-1}. \end{aligned}$$

Then (5) reduces to

$$\mu = \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_0 - \tilde{W}_0 m - Q m_n G G_d \\ G_0 \end{bmatrix} \right\|_\infty$$

Now let

$$Q_1 := Q G G_d.$$

As before, under mild conditions on the plant and the weighting functions, it can be shown that

$$\mu = \inf_{Q_1 \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_0 - \tilde{W}_0 m - Q_1 m_n \\ G_0 \end{bmatrix} \right\|_\infty. \quad (6)$$

It is interesting to note that if G_d is outer (i.e., the transfer function P has no poles in the open right half plane) (3) reduces to

$$\mu = \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} W_0 - m \tilde{Q} \\ -W_1 W_2 G^{-1} \end{bmatrix} \right\|_\infty$$

where

$$\begin{aligned} \tilde{Q} &:= G(X + Q G_d) G_n \\ m &:= m_n b_1. \end{aligned}$$

This is the standard form of a 2-block problem to which the previous skew Toeplitz theory applies immediately. A similar remark applies to the 2-block problem (4) considered above.

Consider the standard form (6). Let $S : H^2 \rightarrow H^2$ denote the unilateral shift, $H(m_v) := H^2 \ominus m_v H^2$ and let $P_{H(m_v)}$ be the orthogonal projection onto $H(m_v)$. Then it follows from the Commutant Lifting Theorem that $\mu = \|A\|$ where $A : H^2 \rightarrow H(m_v) \oplus H^2$ is defined by

$$A := \begin{bmatrix} P_{H(m_v)} (W_0(S) - \dot{W}_0(S)m(S)) \\ G_0(S) \end{bmatrix}. \quad (7)$$

In the next section we will develop an approach to computing the singular values and vectors of operators of the form (7).

To conclude this section we will now compute the essential norm (see [4]) of the operator A , which will be denoted by $\|A\|_e$. For the proof see [9].

Proposition 1. For an operator A defined as in (7)

$$\|A\|_e = \max\{\alpha, \beta\}$$

where

$$\alpha := \max\left\{\left\| \begin{bmatrix} W_0(\zeta) \\ G_0(\zeta) \end{bmatrix} \right\| : \zeta \text{ is an essential singularity of } m\right\}$$

$$\beta := \|G_0\|_\infty.$$

Remark. The proof of Proposition 1 given in [9] involves showing that operators A of the form (7) are finite rank perturbations of the type of operator studied in [3], [4], [8], [10], derived from the compression of a rational function. This fact is the key observation in our solution of the mixed sensitivity problem for unstable distributed plants.

3 Singular values of 2-block operator

Let the operator A be defined as in (7) where $W_0, \dot{W}_0, G_0 \in RH^\infty$, $m \in \tilde{H}^\infty$ is inner (arbitrary), $m_v = m_d m$ and $m_d \in \tilde{H}^\infty$ is a finite Blaschke product. We wish to find $\rho \geq 0$ and $0 \neq y \in H^2$ such that

$$(A^* A - \rho^2 I) y = 0. \quad (8)$$

From (7), this is equivalent to

$$\left\{ (W_0(S)^* - \dot{W}_0(S)^* m(S)^*) P_{H(m_v)} (W_0(S) - \dot{W}_0(S)m(S)) + G_0(S)^* G_0(S) - \rho^2 I \right\} y = 0. \quad (9)$$

Now write

$$W_0 = \frac{B}{K}, \quad \dot{W}_0 = \frac{C}{K}, \quad G_0 = \frac{D}{K}$$

where B, C, D and K are real polynomials. Then (9) holds for some $0 \neq y \in H^2$ if and only if

$$Rx := \left\{ (B(S)^* - C(S)^* m(S)^*) P_{H(m_v)} (B(S) - C(S)m(S)) + D(S)^* D(S) - \rho^2 K(S)^* K(S) \right\} x = 0 \quad (10)$$

holds for some $0 \neq x \in H^2$.

In order to solve (10) for ρ and x we will need to expand the operator R . First note that

$$\begin{aligned} P_{H(m_v)} (B(S) - C(S)m(S)) x &= (B - Cm)x - m_v P_{H^2} \tilde{m}_v (B - Cm)x \\ &= (B - Cm)x + m m_d P_{H^2} \tilde{m}_d Cx - m_v P_{H^2} \tilde{m}_v Bx \\ &= Bx - m P_{H(m_d)} Cx - m_v P_{H^2} \tilde{m}_v Bx. \end{aligned} \quad (11)$$

Thus

$$\begin{aligned} m(S)^* P_{H(m_v)} (B(S) - C(S)m(S)) x &= P_{H^2} \tilde{m} (Bx - m P_{H(m_d)} Cx - m_d P_{H^2} \tilde{m}_v Bx) \\ &= P_{H^2} \tilde{m} Bx - P_{H(m_d)} Cx - m_d P_{H^2} \tilde{m}_v Bx. \end{aligned} \quad (12)$$

From (11) and (12) we obtain

$$\begin{aligned} Rx &= \{D(S)^* D(S) - \rho^2 K(S)^* K(S) + B(S)^* B(S)\} x \\ &\quad + (C(S)^* - B(S)^* m(S)) (P_{H(m_d)} Cx + m_d P_{H^2} \tilde{m}_v Bx) \\ &\quad - C(S)^* P_{H^2} \tilde{m} Bx. \end{aligned} \quad (13)$$

We next decompose x as

$$x = u + m_v v \quad (14)$$

where $u \in H(m_v)$ and $v \in H^2$. Observing that

$$\begin{aligned} H(m_v) &= (H^2 \ominus mH^2) \oplus (mH^2 \ominus m_v H^2) \\ &= (H^2 \ominus mH^2) \oplus m (H^2 \ominus m_d H^2) \\ &= H(m) \oplus mH(m_d) \end{aligned}$$

we can also write

$$u = p + m q \quad (15)$$

where $p \in H(m)$ and $q \in H(m_d)$. Using (14) and (15) we obtain

$$P_{H(m_d)} Cx = P_{H(m_d)} Cu, \quad (16)$$

$$\begin{aligned} P_{H^2} B \tilde{m}_v x &= P_{H^2} B \tilde{m}_v u + Bv \\ &= P_{H^2} B \tilde{m}_v p + P_{H^2} B \tilde{m}_v q + Bv, \end{aligned} \quad (17)$$

$$\begin{aligned} P_{H^2} B \tilde{m} x &= P_{H^2} B \tilde{m} u + B m_d v \\ &= P_{H^2} B \tilde{m} p + Bq + B m_d v. \end{aligned} \quad (18)$$

Substituting (16), (17) and (18) into (13) gives

$$\begin{aligned} Rx &= \{D(S)^* D(S) - \rho^2 K(S)^* K(S) + B(S)^* B(S)\} u \\ &\quad + \{D(S)^* D(S) - \rho^2 K(S)^* K(S)\} m_v v \\ &\quad + (C(S)^* - B(S)^* m(S)) (P_{H(m_d)} Cu \\ &\quad + m_d P_{H^2} B \tilde{m}_v p + m_d P_{H^2} B \tilde{m}_v q) \\ &\quad - C(S)^* (P_{H^2} B \tilde{m} p + Bq). \end{aligned} \quad (19)$$

To understand the right hand side of (19) we will need some further notation. Since $S^* S = I$ and all polynomials have real coefficients we can write

$$\begin{aligned} \{D(S)^* D(S) - \rho^2 K(S)^* K(S) + B(S)^* B(S)\} &=: P(S, S^*) \\ &=: P_{-n} S^n + \dots + P_0 + \dots + P_n S^n \end{aligned}$$

where in fact $P_i = P_{-i}$. Similarly we can write

$$\begin{aligned} \{D(S)^* D(S) - \rho^2 K(S)^* K(S)\} &=: Q(S, S^*) \\ &=: Q_{-n} S^n + \dots + Q_0 + \dots + Q_n S^n \end{aligned}$$

where $Q_i = Q_{-i}$. We thus obtain

$$P(S, S^*) u = P(z, z^{-1}) u - \sum_{i=1}^n P_{-i} \sum_{j=0}^{i-1} z^{-i+j} \gamma_j \quad (20)$$

and

$$Q(S, S^*) m_v v = Q(z, z^{-1}) m_v v - \sum_{i=1}^n Q_{-i} \sum_{j=0}^{i-1} z^{-i+j} \sum_{k=0}^j (m_v)_{j-k} \delta_j \quad (21)$$

where

$$u =: \gamma_0 + z\gamma_1 + z^2\gamma_2 + \dots \quad (22)$$

$$v =: \delta_0 + z\delta_1 + z^2\delta_2 + \dots \quad (23)$$

This deals with the first two terms on the right hand side of (19). We now proceed to show that the remaining two terms can be expressed as a linear combination of the γ_i 's, δ_i 's and certain other coefficients which will be introduced below.

Since $p \in H(m)$ it follows that $\bar{m}p \in H^{2,1}$. Hence we can write

$$\bar{m}p = \gamma_{-1}z^{-1} + \gamma_{-2}z^{-2} + \dots \quad (24)$$

Straightforward calculation then shows that

$$P_{H^2} B \bar{m} p = \sum_{i=1}^n \left(\sum_{j=i}^n B_j z^{j-i} \right) \gamma_{-i} \quad (25)$$

$$P_{H^2} \bar{m}_d B \bar{m} p = \sum_{i=1}^n \left(\sum_{j=i}^n \left(\sum_{r=0}^{n-j} (\bar{m}_d)_r B_{j+r} \right) z^{j-i} \right) \gamma_{-i}. \quad (26)$$

We can then obtain expressions involving the γ_{-i} 's for

$$C(S)^* P_{H^2} B \bar{m} p \quad (27)$$

and

$$(C(S)^* - B(S)^* m(S)) m_d P_{H^2} \bar{m}_d B \bar{m} p. \quad (28)$$

These expressions are written out explicitly in [9].

Now let

$$m_d =: \prod_{i=1}^l \frac{z - a_i}{1 - \bar{a}_i z}$$

where we assume the a_i 's are distinct and non-zero. Then

$$f_i(z) := \frac{1}{1 - \bar{a}_i z}$$

$i = 1, \dots, \ell$ is a basis for $H(m_d)$. We can therefore write

$$q = \sum_{i=1}^l \alpha_i f_i(z) \quad (29)$$

for some α_i which leads to an expressions involving the α_i 's for

$$C(S)^* B(S) q \quad (30)$$

(see [9]). A routine calculation shows that

$$P_{H^2} B \bar{m}_d q = - \sum_{i=1}^l \alpha_i \sum_{j=1}^n c_{ij} \left(\sum_{r=j}^n B_r z^{r-j} \right) \quad (31)$$

where

$$c_{ij} := \sum_{s=0}^{j-1} (\bar{m}_d)_s (\bar{a}_i)^{s-j}$$

from which we can obtain an expression (see [9]) involving the α_i 's for

$$(C(S)^* - B(S)^* m(S)) m_d P_{H^2} B \bar{m}_d q. \quad (32)$$

To evaluate the one remaining term in (19) we observe that, for any $g \in H^2$,

$$P_{H(m_d)} g = \sum_{i,j=1}^l g(a_i) (\Lambda^{-1})_{ij} f_j(z)$$

where

$$\Lambda := \left(\frac{1}{1 - \bar{a}_j a_i} \right)_{i,j}. \quad (33)$$

Thus

$$P_{H(m_d)} C u = \sum_{i,j=1}^l \beta_i C(a_i) (\Lambda^{-1})_{ij} f_j(z) \quad (34)$$

where

$$\beta_i := u(a_i) \quad (35)$$

for $i = 1, \dots, \ell$. Substituting (20), (21), (27), (28), (30), (32) and (34) into (19) gives an expression of the form

$$R x = P(z, z^{-1}) u + Q(z, z^{-1}) m_v v + T(z) \Phi \quad (36)$$

where $T(z)$ is a known vector of length $3n + 2\ell$ with entries in $z^{-n} H^2$ and

$$\Phi^T := [\gamma_{-n}, \dots, \gamma_{n-1}, \delta_0, \dots, \delta_{n-1}, \alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_\ell].$$

To find the singular values and vectors of A we need to find $\rho \geq 0$ and $u \in H(m_u)$ and $v \in H^2$ (not both zero) such that $R x = 0$. Our approach will be to use (36) to find an appropriate non-zero Φ and then to solve for the corresponding u and v . Assuming ρ is a singular value of A we have

$$0 = P(z, z^{-1}) u + Q(z, z^{-1}) m_v v + T(z) \Phi \quad (37)$$

for some $u \in H(m_u)$ and $v \in H^2$ (not both zero). This is equivalent to the projections of (37) onto $H(m_u)$ and $m_v H^2$ both being zero. Taking first the projection onto $m_v H^2$ we obtain

$$\begin{aligned} 0 &= P_{H^2} \bar{m}_v P(z, z^{-1}) u + Q(z, z^{-1}) v \\ &\quad - \sum_{i=1}^n Q_i z^{-i} \sum_{j=0}^{i-1} z^j \delta_j + P_{H^2} \bar{m}_v T(z) \Phi \\ &=: Q(z, z^{-1}) v + T_v(z) \Phi. \end{aligned} \quad (38)$$

We observe from Proposition 1 that, for $\rho > \|A\|_e$, $Q(z, z^{-1})$ has no roots on the unit circle. For simplicity we make the following assumption, which can in fact be removed as in [6].

Assumption 1. The choice of ρ is such that the n roots z_1, \dots, z_n of $Q(z, z^{-1})$ inside the unit disc are distinct.

Since v is analytic inside the disc, it follows that, for ρ a singular value of A ,

$$T_v(z_i) \Phi = 0 \quad (39)$$

for $i = 1, \dots, n$. Next, taking the projection of (37) onto $H(m_u)$ we get

$$\begin{aligned} 0 &= P_{H(m_u)} P(z, z^{-1}) u + P_{H(m_u)} Q(z, z^{-1}) m_v v + P_{H(m_u)} T(z) \Phi \\ &= P(z, z^{-1}) u - \sum_{i=1}^n P_i z^{-i} \sum_{j=0}^{i-1} z^j \gamma_j - m_v P_{H^2} \bar{m}_v P(z, z^{-1}) u \\ &\quad + P_{H(m_u)} Q(z, z^{-1}) m_v v + P_{H(m_u)} T(z) \Phi \\ &=: P(z, z^{-1}) u + T_u(z) \Phi. \end{aligned} \quad (40)$$

We observe from Proposition 1 that, for $\rho > \|A\|_e$, none of the roots of $P(z, z^{-1})$ coincide with essential singularities of m . Again, for simplicity we make the following genericity assumption. Let z_{n+1}, \dots, z_{3n} be the roots of $P(z, z^{-1})$

where the first r roots are those lying in the closed unit disc.

Assumption 2. The choice of ρ is such that z_{n+1}, \dots, z_{3n} are distinct and non-zero.

Since u is analytic inside the disc, it follows that, for ρ a singular value of A ,

$$T_u(z_i)\Phi = 0 \quad (41)$$

for $i = n+1, \dots, n+r$. Now note that $\bar{m}_v u \in H^{2+}$ and so is analytic outside the unit disc. It follows that, for ρ a singular value of A ,

$$\bar{m}_v(z_i^{-1})T_u(z_i)\Phi = 0 \quad (42)$$

for $i = n+r+1, \dots, 3n$.

Notice that, if we are given a $\Phi \neq 0$ satisfying (39), (41) and (42) (which amounts to $3n$ equations in $3n+2\ell$ unknowns), we can find vectors $v \in H^2$ and $u \in H(m_v)$ such that (38) and (40) are satisfied. We need to derive additional equations so that u and v satisfy (29) and (35). Taking the projection of (40) onto mH^2 we obtain

$$\begin{aligned} 0 &= P_{H^2} \bar{m} P(z, z^{-1})(p + mq) + P_{H^2} \bar{m} T_u(z)\Phi \\ &= P_{H^2} \bar{m} P(z, z^{-1})p + P(z, z^{-1})q \\ &\quad - \sum_{i=1}^n P_i z^{-i} \sum_{j=0}^{i-1} z^j q_j + P_{H^2} \bar{m} T_u(z)\Phi \\ &= P(z, z^{-1})q + T_q(z)\Phi. \end{aligned} \quad (43)$$

Then, if ρ is a singular value of A ,

$$0 = P(a_i, a_i^{-1}) \sum_{i=1}^{\ell} \alpha_i f_i(a_i) + T_q(a_i)\Phi \quad (44)$$

for $i = 1, \dots, \ell$. To complete our set of conditions on Φ we obtain from (40) that

$$0 = P(a_i, a_i^{-1})\beta_i + T_u(a_i)\Phi \quad (45)$$

for $i = 1, \dots, \ell$. Finally we need one more assumption.

Assumption 3. The roots z_{n+1}, \dots, z_{n+r} of $P(z, z^{-1})$ inside the unit disc are disjoint from a_1, \dots, a_ℓ .

Theorem 1. Let assumptions 1, 2 and 3 hold. Then $\rho > \|A\|_\infty$ is a singular value of A if and only if some $0 \neq \Phi$ satisfies the system of $3n+2\ell$ equations (39), (41), (42), (44) and (45). Moreover, the corresponding singular vector $y \in H^2$ is given by $K(z)^{-1}(u + m_v v)$ where u and v are determined from (40) and (38).

Proof. The necessity part is immediate from the above derivation. Note that if $0 \neq x$ then either $0 \neq u$ or $0 \neq v$. Thus from (40) or (38) it follows that $0 \neq \Phi$. To show sufficiency let us assume that we have found some vector $0 \neq \Phi'$ satisfying the $3n+2\ell$ equations. Then we can compute a $u \in H(m_v)$ and a $v \in H^2$ from (40) and (38) which are candidates to make up a singular vector. From this u and v we can compute the coefficients $\alpha_i, \beta_i, \gamma_i$ and δ_i and form a vector Φ . To complete the proof we need to show that $\Phi = \Phi'$. This will establish, using (36), that $Rx = 0$. This will also prove that $0 \neq \Phi$ implies that $0 \neq x$. The details of this derivation are given in [9]. \square

The above theorem gives us a way of finding the singular values and vectors of the operator A . The system of $3n+2\ell$ equations (39), (41), (42), (44) and (45) constitute the so called *singular system* [4]. In [9] the singular system is written out in explicit matricial form. The computation of the *maximal* singular value and the associated singular vectors of A then allows us to find the optimal performance μ of our original control problem and the corresponding

optimal compensator.

4 Example

In this section we give a simple example to illustrate the theory described in the previous sections. We apply all the above computations to an unweighted mixed sensitivity minimization problem. In order to elucidate our methods, we will explicitly work through the required computations step by step.

Consider a plant $P(z) = m(z)/m_d(z)$, where m is arbitrary inner (possibly infinite dimensional) and m_d is a first order Blaschke function:

$$m_d(z) = \frac{z-a}{1-az}$$

with $a \in D$ real and $m(a)$ real. The Bezout identity for this system is

$$Xm + Ym_d = 1,$$

so we can choose $X(z) = 1/m(a)$, constant. In this case the sensitivity and the complementary sensitivity are

$$S(z) = 1 - m(z)/m(a) - m(z)m_d(z)Q(z),$$

$$1 - S(z) = m(z)/m(a) + m(z)m_d(z)Q(z),$$

where Q is the free parameter coming from the Youla parametrization of all stabilizing controllers. In the unweighted mixed sensitivity minimization problem we want to find

$$\mu = \inf_{Q \in H^\infty} \left\| \begin{bmatrix} 1 - m(z)/m(a) \\ m(z)/m(a) \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} m_v(z)Q \right\|_\infty$$

where $m_v(z) = m(z)m_d(z)$. By employing inner/outer factorizations for the constant matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ the above can be reduced to

$$\mu = \inf_{Q \in H^\infty} \left\| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 - 2m(z)/m(a) & -m_v(z)Q \\ 1 & 1 \end{bmatrix} \right\|_\infty.$$

Therefore

$$\mu = \sqrt{\frac{1 + \mu_1^2}{2}},$$

where

$$\mu_1 = \inf_{Q \in H^\infty} \|1 - 2m(z)/m(a) - m_v(z)Q\|_\infty.$$

So the problem is reduced to computing μ_1 , and the corresponding optimal interpolant. A lower bound for μ_1 can be computed by putting $z = a$ in the above equation, and an upper bound can be computed by choosing, say, $Q = 0$, i.e.,

$$1 \leq \mu_1 \leq \|1 - 2m(z)/m(a)\|_\infty.$$

By the Commutant Lifting Theorem we have that $\mu_1 = \|1 - 2m(T)/m(a)\|$, with $T = P_{H(m_v)} S|_{H(m_v)}$. To compute the norm we form the singular value singular vector equation

$$(\rho^2 I - (I - 2m(T)^*/m(a))(I - 2m(T)/m(a)))u = 0 \quad (46)$$

where ρ^2 is a singular value with corresponding singular vector $u \in H(m_v)$. Now we decompose u as $u = p + mq$, where $p \in H(m)$, and $q \in H(m_d)$.

We know the action of $m(T)^*$ and $m(T)$ on u :

$$m(T)^*u = q(z), \quad m(T)u = m(z)P_{H(m_d)}u.$$

We can now write the equation (46) as follows. First note that $H(m_d)$ is one dimensional and has a basis $f(z) = \frac{1}{1-az}$, so $q(z) = \alpha f(z)$ for some constant α , and moreover

$$P_{H(m_d)}u = \beta(1-a^2)f(z)$$

where $\beta := u(a)$ is a constant. We then have that (46) is equivalent to

$$(\rho^2 - 1)u = 4\beta \frac{1-a^2}{m(a)^2} f(z) - 2\beta \frac{1-a^2}{m(a)} m(z)f(z) - 2\alpha \frac{1}{m(a)} f(z). \quad (47)$$

Note that in this case we have $n = 0$ and $\ell = 1$. Hence the number of linearly independent equations that we obtain is $3n + 2\ell = 2$. Evaluating (47) at $z = a$ we obtain one of the equations as

$$(\rho^2 - 1)\beta = 4\beta \frac{1}{m(a)^2} - 2\beta - 2\alpha \frac{1}{m(a)} \frac{1}{1-a^2}. \quad (48)$$

The other equation is obtained by taking the orthogonal projection of (47) onto mH^2 . After simplifications this can be found to be equivalent to

$$2\beta \frac{1-a^2}{m(a)} = (\rho^2 - 1)\alpha. \quad (49)$$

Then μ_1 is the largest value of $\rho \in [1, \|1 - 2m(z)/m(a)\|_\infty]$ satisfying (48) and (49) for some nonzero constants α and β . This can easily be computed from (48) and (49), and the final answer is

$$\mu_1^2 = \left(\frac{2}{m(a)^2} - 1\right) + \frac{2}{m(a)^2} \sqrt{1 - m(a)^2}.$$

Consequently, for this example the optimal mixed sensitivity performance level $\mu = \sqrt{(1 + \mu_1^2)/2}$ can be computed as

$$\mu = \frac{1}{|m(a)|} \sqrt{1 + \sqrt{1 - m(a)^2}}.$$

The optimal controller can be found by finding a nonzero α and β satisfying (48) and (49), and then constructing the singular vector u from these α and β . The vector u then gives the optimal controller going back from the Commutant Lifting Theorem and the Youla parametrization.

An important particular case of the above example is a plant (in continuous time) with a delay and one unstable pole:

$$P(s) = e^{-hs} \frac{\sigma s + 1}{\sigma s - 1}.$$

After transforming the data to the unit disc with the conformal map $z = \frac{s-1}{s+1}$, we find that

$$m(z) = e^{h \frac{z+1}{z-1}}, \quad m_d(z) = \frac{z-a}{1-az},$$

with $a = (1 - \sigma)/(1 + \sigma)$. Then $m(a) = e^{-h/\sigma}$ and hence

$$\mu = e^{h/\sigma} \sqrt{1 + \sqrt{1 - e^{-2h/\sigma}}}.$$

It is interesting to note that as $h \rightarrow \infty$, and/or $\sigma \rightarrow 0$, the best achievable performance increases exponentially, as expected.

5 Concluding remarks

In this paper we have extended the skew Toeplitz theory developed in [1], [3], [4], [5], [6], [8], [10] for stable distributed systems to plants which have finitely many unstable poles. We have assumed that the stable part of the system has an arbitrary inner part and a rational outer part.

The singular system of $3n + 2\ell$ equations obtained in this paper (39), (41), (42), (44) and (45) for the computation of the optimal performance μ and the corresponding optimal compensator is written out in explicit matricial form in [9] and is easily implementable on the computer. We have computed in the paper (by hand) the optimal performance for an unweighted mixed sensitivity problem. These methods have already been employed for the design of an optimal compensator in a flexible beam problem [7].

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